

TOWARDS COVARIANT MATRIX THEORY

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We review an approach towards a covariant formulation of Matrix theory based on a discretization of the 11d membrane. Higher dimensional algebraic structures, such as the quantum triple Nambu bracket, naturally appear in this approach. We also discuss a novel geometric understanding of the space-time uncertainty relation which points towards a more geometric formulation of Matrix theory.

1 Introduction: Matrix vs. Membrane

Matrix theory [1] is conjectured to be M-theory in the infinite momentum frame. In this article we review an approach [2] towards a covariant formulation of Matrix theory based on a suitable discretization of the 11-dimensional membrane [3]. Recall that Matrix theory uses $N \times N$ Hermitian matrices to represent the transverse coordinates (and their super-partners) of N $D0$ -branes X^i ($i = 1, 2, \dots, 9$) [4]. The statistics of $D0$ -branes are encoded in the $U(N)$ gauge symmetry ($t = X^+ = \text{light-cone time}$) $X^i \rightarrow UX^iU^{-1}$. The number N of $D0$ -branes is connected to the longitudinal momentum P^+ in the light-like direction by $P^+ = \frac{N}{R}$ where $R = g_s \ell_s$ is the compactified radius in the X^- direction.

What should be the structure of a covariant Matrix theory? We expect at least: a) a generalization of the matrix algebra and the emergence of higher symmetry, b) the appearance of R (and perhaps N) as dynamical variables, c) a formulation of holography [5] from the “bulk” point of view.

At present, the only guiding principle in trying to search for such a structure is that a covariant version of Matrix theory should reduce to Matrix theory if the light-cone gauge is chosen, and that it should possess 11-dimensional (super) Poincaré invariance in the $R \rightarrow \infty$ limit. We also expect the covariant version of Matrix theory to describe many-body interactions of 11-dimensional (super)gravitons. Time is only globally defined in Matrix theory, so its covariant version should be a quantum mechanical theory invariant under world-line reparametrizations [6]. Finally, the elusive transverse five-brane should be explicitly present in such a covariant formulation.

In attempting to formulate a covariant version of Matrix theory, it is tempting to try to discretize the 11-dimensional world-volume membrane theory. This is natural from the point of view of the correspondence between 2-dimensional area preserving diffeomorphisms (APD) of the light-cone membrane and the Goldstone-Hoppe large N limit [7] of the $U(N)$ symmetry of Matrix theory. In particular, the membrane world volume is invariant under 3-dimensional volume preserving diffeomorphisms (VPD). Thus one might expect that a discretized membrane theory should be formulated in terms of discretized VPD [2]. We are therefore lead to the following pictorial correspondence

$$\begin{array}{ccc}
\text{matrices/commutators} & \leftrightarrow & \text{2D surface} \\
U(N) & \leftrightarrow & \text{2D APD} \\
\\
& \Downarrow & \\
\text{triple Nambu bracket} & \leftrightarrow & \text{3D volume} \\
? & \leftrightarrow & \text{3D VPD}
\end{array}$$

since the two indices of $N \times N$ matrices in Matrix theory simply correspond to the discretized Fourier indices on the membrane, and the triple Nambu bracket generates 3-dimensional VPD.

In this article we first review the formal quantization problem of 3-dimensional VPD, which underlies the covariantization approach presented in [2], and then discuss its physical content embodied in the space-time uncertainty relation [8].

2 Classical and Quantum Nambu Brackets

Consider a three-dimensional space parametrized by $\{x^i\}$. The three-dimensional VPD on this space are described by a differentiable map $x^i \rightarrow y^i(x)$ such that $\{y^1, y^2, y^3\} = 1$ where, by definition,

$$\{A, B, C\} \equiv \epsilon^{ijk} \partial_i A \partial_j B \partial_k C \quad (1)$$

is the Nambu triple bracket [9], which satisfies [10, 11, 12]

1. Skew-symmetry

$$\{A_1, A_2, A_3\} = (-1)^{\epsilon(p)} \{A_{p(1)}, A_{p(2)}, A_{p(3)}\}, \quad (2)$$

where $p(i)$ is the permutation of indices and $\epsilon(p)$ is the parity of the permutation,

2. Derivation

$$\{A_1 A_2, A_3, A_4\} = A_1 \{A_2, A_3, A_4\} + \{A_1, A_3, A_4\} A_2, \quad (3)$$

3. Fundamental Identity (FI-1) [11, 12]

$$\begin{aligned} & \{\{A_1, A_2, A_3\}, A_4, A_5\} + \{A_3, \{A_1, A_2, A_4\}, A_5\} \\ & + \{A_3, A_4, \{A_1, A_2, A_5\}\} = \{A_1, A_2, \{A_3, A_4, A_5\}\}. \end{aligned} \quad (4)$$

The three-dimensional VPD involve two independent functions. Let these functions be denoted by f and g . The infinitesimal three-dimensional VPD generator is then given as

$$D(f, g) \equiv \epsilon^{ijk} \partial_i f \partial_j g \partial_k \equiv D^k(f, g) \partial_k. \quad (5)$$

The volume-preserving property is nothing but the identity $\partial_k(\epsilon^{ijk} \partial_i f \partial_j g) = 0$. Given an arbitrary scalar function $X(x^i)$, the three-dimensional VPD act as

$$D(f, g)X = \{f, g, X\}. \quad (6)$$

Apart from the issue of global definition of the functions f and g , we can represent an arbitrary infinitesimal volume-preserving diffeomorphism in this form. On the other hand, if the base three-dimensional space $\{x^i\}$ is mapped into a flat Euclidean target space of dimension $d + 1$ whose coordinates are X^α ($\alpha = 0, 1, 2, \dots, d$), the induced infinitesimal volume element is

$$d\sigma \equiv \sqrt{\{X^\alpha, X^\beta, X^\gamma\}^2} dx^1 dx^2 dx^3. \quad (7)$$

The volume element is of course invariant under the general three-dimensional diffeomorphisms. The triple product $\{X^\alpha, X^\beta, X^\gamma\}$ is also “invariant” under the VPD. More precisely, it transforms as a scalar. Namely,

$$\{Y^\alpha, Y^\beta, Y^\gamma\} - \{X^\alpha, X^\beta, X^\gamma\} = \epsilon D(f, g) \{X^\alpha, X^\beta, X^\gamma\} + O(\epsilon^2) \quad (8)$$

for $Y = X + \epsilon D(f, g)X$. This is due to the Fundamental Identity FI-1 which shows that the operator $D(f, g)$ acts as a derivation within the Nambu bracket. For fixed f and g , we can define a finite transformation by

$$X(t) \equiv \exp(tD(f, g)) \rightarrow X = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{f, g, \{f, g, \{\dots, \{f, g, \{f, g, X\}\}\dots\}\}\} \quad (9)$$

which satisfies the Nambu “equation of motion” [9]

$$\frac{d}{dt}X(t) = \{f, g, X(t)\}. \quad (10)$$

The Nambu bracket is preserved under this evolution equation.

Notice that in the case of the usual Poisson structure, the algebra of two-dimensional area preserving diffeomorphisms is given by

$$[D(f_1), D(f_2)] = D(f_3) \quad (11)$$

where $f_3 = \{f_1, f_2\}$ and $D(f)X = \{f, X\}$. It turns out that the three-dimensional analogue [2] of the commutator algebra

$$D(A_{[1})D(A_2]) = D(\{A_1, A_2\}) \quad (12)$$

can be written using the quantum triple Nambu commutator [9]

$$[A, B, C]_N \equiv ABC - ACB + BCA - BAC + CAB - CBA \quad (13)$$

as follows [2]

$$D(A_{[1}, A_2)D(A_{3]N}, B) = 2D(\{A_1, A_2, A_3\}, B), \quad (14)$$

or equivalently

$$D(B_{[1}, B_2)D(A_{[1}, A_2)D(A_{3]N}, B_{3]N}) = 4D(\{A_1, A_2, A_3\}, \{B_1, B_2, B_3\}). \quad (15)$$

Both relations are equivalent to the Fundamental Identity. This result suggests that there is a new kind of symmetry based on a new composition law whose infinitesimal algebra is given by the triple commutator (13). It was conjectured in [2] that this symmetry is related to the gauge transformations that are not of the Yang-Mills type.

We turn now to the properties of the quantum Nambu bracket. What do we mean by a quantum triple Nambu bracket? In general we want an object $[F, G, W]$ which satisfies the properties analogous to the classical Nambu bracket $\{f, g, w\}$ as listed above. (Here f, g, w are functions of three variables, and the nature of F, G, W is left open for the moment.) Thus $[F, G, W]$ is expected to satisfy [10, 11, 12]

1. Skew-symmetry

$$[A_1, A_2, A_3] = (-1)^{\epsilon(p)}[A_{p(1)}, A_{p(2)}, A_{p(3)}], \quad (16)$$

2. Derivation

$$[A_1 A_2, A_3, A_4] = A_1 [A_2, A_3, A_4] + [A_1, A_3, A_4] A_2, \quad (17)$$

3. Fundamental Identity (F.I.)

$$\begin{aligned} & [[A_1, A_2, A_3], A_4, A_5] + [A_3, [A_1, A_2, A_4], A_5] \\ & + [A_3, A_4, [A_1, A_2, A_5]] = [A_1, A_2, [A_3, A_4, A_5]]. \end{aligned} \quad (18)$$

An explicit matrix realization of the quantum Nambu bracket, which is skew-symmetric and obeys the Fundamental Identity was given in [2]. The construction proceeds as follows: Define a totally antisymmetric triple bracket of three matrices A, B, C as

$$[A, B, C] \equiv (\text{tr} A)[B, C] + (\text{tr} B)[C, A] + (\text{tr} C)[A, B]. \quad (19)$$

Then $\text{tr}[A, B, C] = 0$, and if $C = 1$, $[A, B, 1] = N[A, B]$, where N is the rank of square matrices. This bracket is obviously skew-symmetric and it can be shown to obey the Fundamental Identity [2]. Consider the following “gauge transformation”

$$\delta A \equiv i[X, Y, A], \quad (20)$$

where the factor i is introduced for Hermitian matrices. This transformation represents an obvious quantum form of the three-dimensional volume preserving diffeomorphisms. By the definition of the triple bracket, the generalized gauge transformation takes the following explicit form

$$\delta A = i((\text{tr} X)Y - (\text{tr} Y)X, A) + (\text{tr} A)[X, Y]. \quad (21)$$

If $\text{tr} A_i = 0, i = 1, \dots, n$, then $\text{tr}(A_1 A_2 \dots A_n)$ is gauge invariant. The gauge transformation (21) indicates that a bosonic Hermitian matrix A can be transformed into a form proportional to the unit $N \times N$ matrix as long as $\text{tr} A \neq 0$. In other words, since the gauge transformation is traceless, one can show that $A \rightarrow \frac{1}{N}(\text{tr} A)\mathbf{1}_{N \times N}$.

The triple quantum Nambu bracket can also be represented in terms of three-index objects or cubic matrices [2]. These objects might be relevant for the description of the five-brane degrees of freedom.

The explicit example of the quantum Nambu bracket presented above does not satisfy the derivation property. Hence, this form of the quantum Nambu bracket does not completely parallel the form of the classical Nambu bracket. For example, the classical triple Nambu bracket of three functions f, g, h of

three variables τ, σ_1, σ_2 can be obviously rewritten as $\{f, g, w\} = \dot{f}\{g, w\} + \dot{g}\{w, f\} + \dot{w}\{f, g\}$ where $\dot{f} = \partial_\tau f$ and $\{f, g\} = \partial_{\sigma_1} f \partial_{\sigma_2} g - \partial_{\sigma_2} f \partial_{\sigma_1} g$. If we try to extrapolate our previous definition of the quantum Nambu bracket in terms of τ -dependent square matrices to $[F, G, W] = \dot{F}[G, W] + \dot{G}[W, F] + \dot{W}[F, G]$ where $\dot{F} = \partial_\tau F$, one can show that such $[F, G, W]$ does not satisfy the Fundamental Identity.

Finally, we note that the formal quantization problem of Nambu brackets was solved using the Zariski deformation quantization, based on factorization of polynomials in several real variables [10]. Unfortunately, a matrix realization of this quantization procedure is not known. Thus a discretized version of the 11-dimensional membrane based on the triple Nambu bracket is still unavailable.

To make progress, one should perhaps consider (in the $N \rightarrow \infty$ limit), instead of Hermitian $N \times N$ matrices, selfadjoint operators on a Hilbert space, as is customary in Connes' quantized calculus, and apply this machinery to the Polyakov action for the 11-dimensional membrane following [13].

3 Space-Time Uncertainty Relation and Geometry of M-theory

In a variety of contexts in perturbative and non-perturbative string theory the following space-time uncertainty relation [8] appears

$$\delta T \delta X \sim \alpha'. \quad (22)$$

Here δT and δX measure the appropriate longitudinal and transverse space-time distances. Although eq. (22) is nothing but the usual energy-time uncertainty relation applied to string theory, in which $\delta E \sim \delta X / l_s^2$, this relation appears in different ways in different aspects of string theory. In perturbative string theory the space-time uncertainty relation stems from conformal symmetry [8]. In non-perturbative string/M theory, the space-time uncertainty relation captures the essential features of the physics of D-branes as well as the properties of holography [5] and the UV/IR relation [14]. Eq. (22) is true in Matrix theory if $X^a \rightarrow \lambda X^a, t \rightarrow \lambda^{-1} t$, provided the longitudinal distance is identified with the global time of Matrix theory and provided the string coupling constant is simultaneously rescaled $g_s \rightarrow \lambda^3 g_s$. The space-time uncertainty relation hence leads readily to the well known characteristic space-time scales in M-theory [15].

The string theory space-time uncertainty relation can also be understood as a limit of a space-time uncertainty relation in M-theory [8]. In Matrix theory eq. (22) can be rewritten as $\delta T \delta X_T \sim l_p^3/R$ where δX_T and δT respectively measure transverse spatial and time directions. Note that the uncertainty for the longitudinal direction in physical processes that involve individual $D0$ -branes is $\delta X_L \sim R$. Thus

$$\delta T \delta X_T \delta X_L \sim l_p^3. \quad (23)$$

This is the space-time uncertainty relation in M-theory. Note that this “triple” relation naturally invokes the form of the triple Nambu bracket, discussed in section 2, thus providing another physical motivation for the covariantization approach of [2].

Here we want to comment on the geometrical structure underlying the space-time uncertainty relation which points towards a more geometric formulation of Matrix theory.

Recall the remarkable fact that the projective Hilbert space \mathcal{P} , defined as the set of rays of the Hilbert space \mathcal{H} , for any quantum mechanical system is a Kähler manifold [16]. States of a quantum mechanical system are represented as points of this manifold. The Schrödinger dynamical evolution is represented by the symplectic flow generated by a Hamiltonian function. The distance along a given curve in the projective Hilbert space is given by the Fubini-Study metric. In particular, the infinitesimal distance defined by the line element of the Fubini-Study metric is proportional to the energy-time uncertainty [16]!

Let us apply this beautiful geometric formulation of quantum mechanics to a very particular quantum mechanics of gravity - Matrix theory. By reinterpreting the space-time uncertainty as the energy-time uncertainty, we have the claim: *the space-time uncertainty in Matrix theory measures the infinitesimal distance along a given curve in the projective Hilbert space of Matrix theory.*

Recall that the Hilbert space of Matrix theory can be represented in terms of block diagonal $N \times N$ Hermitian matrices (at large N), according to [1]. This fact offers a tantalizing possibility that the kinematical set-up as described by the geometric formulation of quantum mechanics [16] can be naturally generalized in the case of quantum mechanics of gravity: the points of the symplectic manifold of that quantum mechanics can be turned into matrices and the Hamiltonian formulation of the Schrödinger evolution can be generalized to incorporate a non-Abelian structure!

Obviously, a geometrical reinterpretation of Matrix theory is called for in view of the geometric meaning of the space-time uncertainty relation. One possible geometric interpretation might be provided by an M-theoretic general-

ization of the geometric formulation of perturbative string theory as presented some years ago in [17]. One particularly striking result of that work was that the equations of motion for the massless fields of the closed string theory were obtained from the existence of a reparametrization invariant symplectic structure on the loop space. This was a geometric reinterpretation of the usual requirement of conformal invariance in perturbative string theory.

It is natural to conjecture that *the equations of motion of the 11-dimensional supergravity follow from the existence of a $U(\infty)$ invariant symplectic structure of the appropriate Kähler manifold associated with the projective Hilbert space of Matrix theory*. This statement would be the Matrix theory analogue of the requirement of conformal invariance in perturbative string theory.

We hope to address these fascinating issues in detail elsewhere.

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